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Coefficients of fractional parentage of a boson system with *F*-spin

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Abstract. Wavefunctions of bosons each with angular momentum l and F-spin $\frac{1}{2}$ are classified according to the group chain $U(2N)\supset(U(N)\supset O(N)\supset O(3))\times SU(2)$. The boson number, seniority, reduced F-spin, total angular momentum, and total F-spin quantum numbers are used to label the wavefunctions. The corresponding coefficients of fractional percentage (CFP) are factorized into F-spin and angular momentum parts. The F-spin parts are calcualted by an analytical formula and the angular momentum parts are calculated by recursion technique. The isoscalar factors of $U(2N)\supset U(N)\supset O(N)$ and the F-spin part of the CFP for the cases $F=F_{\max}$ and $F=F_{\max}-1$ are given explicitly.

1. Introduction

The interacting boson model (IBM) has been successful in describing nuclear structure [1]. In its original version (IBM1), which does not distinguish the proton boson from the neutron boson, the three kinds of nuclear collective motion can be described quite well. After the development of IBM1, version IBM2, which treats the proton boson and neutron boson separately, was proposed. Since then, remarkable progress has been made in both the theory and the agreement between the theoretical predictions and experimental data [2, 3]. Now that the g-boson has been introduced, the well deformed nuclei and higher excited states can be described, and the nuclear octupole deformation is described by considering the f- and p-bosons [4–9].

It is known that the coefficients of fractional parentage (CFP) [10–13] method is one of the most efficient techniques for constructing the IBM wavefunctions. Even though much work has been undertaken in the study of nuclear physics, discussion of the CFP of IBM1 and especially the CFP of IBM2 has been relatively meagre. Since calculations of the CFP are quite time consuming, the working load of calculations for some computer codes in the framework of the IBM is tremendously heavy. Moreover, since CFP tables with large numbers of bosons are not available, calculations with large number of bosons are not possible. These defects have limited the application domain of IBM. Recently, Sun and collaborators have put forward a simple formula to calculate the CFP of IBM1 [14, 15]. A computer code based on the formula has also been set up to determine the CFP of IBM1 [16]. This computer code is quite efficient: it takes only

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20 minutes on the IBM 350 machine to obtain all the CFP for a system including 36 d-bosons. It provides us with a convenient way to describe the nuclear superdeformed states and chaotic behaviour of the boson system.

In fact, as the nuclear physics can be discussed in the framework of the IBM, it is usually recognized that IBM2 has a better microscopic foundation. It is certain that the proton boson and the neutron boson should be treated separately in this case. However, the configuration space is enlarged remarkably. It increases the difficulty of calculation greatly, so that practical applications are limited. In order to restrict the configuration space to a manageable size, it has usually to be truncated. If the wavefunctions are classified by more physical quantum numbers such as seniority or F-spin, the configuration space is easier to truncate physically.

The F-spin, which for a boson system has a role similar to the isospin for fermions has proved to be a good approximate quantum number [17, 18]. However, it was also conjectured that some collective modes corresponding to the mixtures of states with different F-spin values could arise [19]. The discovery of low-lying 1^+ states was believed to be strong support for this conjecture [20]. Since then the question of the purity of symmetry properties with the F-spin value $=F_{max}$ has been studied quite extensively in the vibrational, rotational and γ-unstable rotor limits [21–25]. Nevertheless, most of the real nuclides lie in between the mentioned limits. Therefore, to carry out detailed calculations for such transitional nuclides a phenomenological model with a reasonably truncated model space is certainly needed. To construct n-boson wavefunctions with F-spin, the CFP with F-spin need to be calculated.

In this paper, using the Lie group theory we discuss the classification of the wavefunctions of a boson system with a single angular momentum l and F-spin $\frac{1}{2}$ according to the group chain $U(2N)\supset(U(N)\supset O(N)\supset O(3))\times SU(2)$. These wavefunctions are simply those with a well-defined boson number, seniority, reduced F-spin angular momentum L and total F-spin. The corresponding CFP can be factorized into the F-spin part and the angular momentum part. The F-spin part of the CFP can be given analytically [15], and some of them are given in this paper. A recursion relation of the angular momentum part is presented. This is not only useful to truncation but also helpful to reduce the difficulty of calculation.

2. Classification of wavefunctions

The wavefunctions of a system with bosons of angular momentum l and F-spin $\frac{1}{2}$ can be classified according to the following group chain:

$$U(2N)\supset(U(N)\supset O(N)\supset O(3))\times SU(2) \tag{2.1}$$

where $N=2l+1$. Assuming

$$\begin{aligned} b^{\dagger}_{m\sigma} &= b^{\dagger}_{lm1/2\sigma} \\ b_{m\sigma} &= b_{lm1/2\sigma} \end{aligned} \tag{2.2a}$$

which are the creation and annihilation operators of a boson with angular momentum l, F-spin $\frac{1}{2}$ and Z-components m, σ. Let

$$\tilde{b}_{m\sigma}=(-)^{l+m+1/2+\sigma}b_{-m-\sigma} \tag{2.2b}$$

Table 1. The generators, Casimir operators, IRRP labels and eigenvalues of the Casimir operators for the groups in the group chain (2.1).

Group	Generators	Casimir operators	IRRP labels	Eigenvalues of the Casimir operators
U(2N)	$B^{kt}_{qu}=(b^\dagger\tilde{b})^{kt}_{qu}$	$C_{2U2N}=n_D$ $C_{2U2N}=\Sigma_{kt}B^{kt}\cdot B^{kt}$	n	$n(n+2N-1)$
U(N)	$P^k_q=\sqrt{2}B^{ko}_{qo}$	$C_{2UN}=\Sigma_k P^k\cdot P^k$	$[n_1n_2]$	$n_1(n_1+N-1)+n_2(n_2+N-3)$
O(N)	$P^k_q=\sqrt{2}B^{ko}_{qo}$	$C_{2ON}=\Sigma_{k=\text{odd}}P^k\cdot P^k$	$(\sigma_1\sigma_1)$	$\sigma_1(\sigma_1+N-2)+\sigma_2(\sigma_2+N-4)$
O(3)	$L_q=\sqrt{[l(l+1)N]/3}P^l_q$	$C_{2O3}=\hat{L}\cdot\hat{L}$	L	$L(L+1)$
SU(2)	$\hat{F}_u=\sqrt{N/2}B^{ol}_{ou}$	$C_{2SU2}=\hat{F}\cdot\hat{F}$	F	$F(F+1)$

be the irreducible tensor corresponding to $b_{m\sigma}$. The generators, Casimir operators, the labels of the irreducible representations (IRRPS) and the eigenvalues of the Casimir operators for each subgroup in the group chain (2.1) are given in table 1.

The wavefunctions of the boson system can then be written as

$$\begin{array}{l}|n[n_1n_2]\,\beta(\sigma_1\sigma_2)\alpha L\,F)\\ \mathrm{U}(2N)\mathrm{U}(N)\;\mathrm{O}(N)\;\mathrm{O}(3)\;\mathrm{SU}(2)\end{array} \tag{2.3}$$

where β, α are additional quantum numbers; the reason for including these additional quantum numbers is that the reductions of $\mathrm{U}(N)\supset\mathrm{O}(N)$ and $\mathrm{O}(N)\supset\mathrm{O}(3)$ are not simple. According to the definition of equation (2.3), $|n[n_1n_2]\,\beta(\sigma_1\sigma_2\alpha L\,F\rangle$ satisfies the following relations:

$$\begin{bmatrix}C_{2U2N}\\ C_{2UN}\\ C_{2ON}\\ C_{2O3}\\ C_{2SU2}\end{bmatrix}|n(n_1n_2)\,\beta(\sigma_1\sigma_2)\propto LF\rangle$$

$$=\begin{bmatrix}n(n+2N-1)\\ n_1(n_1+N-1)+n_2(n_2+N-3)\\ \sigma_1(\sigma_1+N-2)+\sigma_2(\sigma_2+N-4)\\ L(L+1)\\ F(F+1).\end{bmatrix}|n[n_1n_2]\,\beta(\sigma_1\sigma_2)\alpha LF\rangle. \tag{2.4}$$

3. The branching rules of the reduction U(2N) ⊃ (U(N) ⊃ O(N) ⊃ O(3)) × SU(2)

3.1. The reduction of U(2N) ⊃ U(N) × SU(2)

The branching rule for this reduction is quite simple. It can be expressed as

$$n=\sum_{n_1,n_2}[n_1n_2]\times F \tag{3.1a}$$

where

$$n=n_1+n_2 \qquad F=\frac{n_1-n_2}{2}. \tag{3.1b}$$

3.2. The reduction of U(N) ⊃ O(N)

By considering the Kronecker product of the IRRPS of group U(*N*)

$$[n_1]\times[n_2]=[n_1+n_2]+[n_1+n_2-1,1]+\ldots+[n_1n_2] \tag{3.2}$$

where

$$n_1\geqslant n_2$$

we get

$$[n_1n_2]=[n_1]\times[n_2]-[n_1+1]\times[n_2-1]. \tag{3.3}$$

In the same way we find the relation between the Kronecker product of the IRRPS of group O(*N*) [26]:

$$(\sigma)\times(\sigma')=(\sigma_1+1)\times(\sigma'-1)+\sum_{\alpha=0}^{\sigma'}(\sigma-\sigma'+\alpha,\alpha). \tag{3.4}$$

Taking the branching rule for the totally symmetric IRRPS of group U(*N*),

$$[n]=(n)+(n-2)+(n-4)+\ldots \tag{3.5}$$

into account, we get the following recurrent relations of the branching rule of the reduction U(*N*) ⊃ O(*N*):

$$\begin{aligned}&[n,n]=[n-2,n-2]+F(n,n)-F(n-1,n-1)\\&[n,n-1]=F(n,n-1)\\&\ldots\\&[n_1n_2]=[n_1-2,n_2]+F(n_1,n_2)\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}&F(n_1,n_2)=\sum_{\beta\alpha}(n_1-\beta+\alpha,\alpha)\\&\beta=n_2,n_2-2,n_2-4,\ldots,\geqslant 0\\&\alpha=\beta,\beta-1,\beta-2,\ldots,0.\end{aligned} \tag{3.7}$$

Using equations (3.6) and (3.7), all of the branching rules of this reduction can be obtained. Some branching rules of the reduction U(*N*) ⊃ O(*N*) are given in table 2.

3.3. The reduction of O(N) ⊃ O(3)

A method to obtain the branching rules for this reduction has been proposed by Wang and Sun [27], and with use of the computer codes [28] all the branching rules were obtained. Here we give the maximum values of the quantum number *L* for the IRRP

Table 2. Reduction of U(N)$\supset$O(N).

$[n]$			n even		
$(n\,0)$	$(n-2\,0)$	$(n-4\,0)$	$(4\,0)$	$(2\,0)$	$(0\,0)$
$[n-1\,1]$					
$(n-2\,0)$	$(n-4\,0)$	$(n-6\,0)$	$(4\,0)$	$(2\,0)$	
$(n-1\,1)$	$(n-3\,1)$	$(n-5\,1)$	$(5\,1)$	$(3\,1)$	$(1\,1)$
$[n-2\,2]$					
$(n-2\,0)$	$(n-4\,0)^2$	$(n-6\,0)^2$	$(4\,0)^2$	$(2\,0)^2$	$(0\,0)$
	$(n-3\,1)$	$(n-5\,1)$	$(5\,1)$	$(3\,1)$	
	$(n-2\,2)$	$(n-4\,2)$	$(6\,2)$	$(4\,2)$	$(2\,2)$
$[n]$			n odd		
$(n\,0)$	$(n-2\,0)$	$(n-4\,0)$	$(5\,0)$	$(3\,0)$	$(1\,0)$
$[n-1\,1]$					
$(n-2\,0)$	$(n-4\,0)$	$(n-6\,0)$ ---	$(5\,0)$	$(3\,0)$	$(1\,0)$
$(n-1\,1)$	$(n-3\,1)$	$(n-5\,1)$ ---	$(6\,1)$	$(4\,1)$	$(2\,1)$
$[n-2\,2]$					
$(n-2\,0)$	$(n-4\,0)^2$	$(n-6\,0)^2$ ---	$(5\,0)^2$	$(3\,0)^2$	$(1\,0)$
	$(n-3\,1)$	$(n-5\,1)$ ---	$(6\,1)$	$(4\,1)$	$(2\,1)$
	$(n-2\,2)$	$(n-4\,2)$ ---	$(7\,2)$	$(5\,2)$	$(3\,2)$

$(\sigma_1\sigma_2)$ of the group O(N):

$$L_{\max}=(\sigma_1+\sigma_2)l-\sigma_2. \tag{3.8}$$

It is well known that the reduction of the IRRPS of a group is rather complicated, so that the reduction should be checked for correctness. An efficient checking measure is to compare the dimensions of the IRRPS. The dimensions of the IRRPS of groups U($2N$), U(N) and O(N) are given by the following formulae [26]:

$$d(n)=\frac{(n+2N-1)!}{n!(2N-1)!}$$

$$d[n_1n_2]=\frac{(n_1+N-1)!}{(n_1+1)!\,(N-1)!}\,\frac{(n_2+N-2)!}{n_2!\,(N-2)!}\,(n_1-n_2+1) \tag{3.9}$$

$$d[\sigma_1\sigma_2]=\frac{(\sigma_1+N-4)!}{(\sigma_1+1)!\,(N-2)!}\,\frac{(\sigma_2+N-5)!}{\sigma_2!(N-4)!}\,(2\sigma_1+N-2)$$

$$\times(2\sigma_2+N-4)(\sigma_1+\sigma_2+N-3)(\sigma_1-\sigma_2+1).$$

4. The wavefunctions $|n[n_1n_2]\,\beta(\sigma_1\sigma_2)\,\alpha\,L\,F\rangle$

From the above discussion we see that, in order to label the wavefunctions completely, seven parameters are needed, which are $n_1, n_2, \beta, \sigma_1, \sigma_2, \alpha, L$. Using the boson operators (2.2*a*) and (2.2*b*), we obtain

$$C_{2U2N}=\hat{n}_b(\hat{n}_b+2N-1) \tag{4.1}$$

$$C_{2UN}=\tfrac{1}{2}\hat{n}_b(\hat{n}_b+2N-4)+2\hat{F}\cdot\hat{F} \tag{4.2}$$

$$C_{2ON}=\tfrac{1}{2}\hat{n}_b(\hat{n}_b+2N-6)+2\hat{F}\cdot\hat{F}-2p^\dagger\cdot\tilde{p} \tag{4.3}$$

where

$$\begin{aligned} F_u&=\sqrt{\frac{N}{2}}\,(b^\dagger\tilde{b})^{0\,1}_{0\,u} \\ p_u^\dagger&=\sqrt{\frac{N}{2}}\,(b^\dagger b^\dagger)^{0\,1}_{0\,u} \\ \tilde{p}_u&=\sqrt{\frac{N}{2}}\,(\tilde{b}\tilde{b})^{0\,1}_{0\,u} \end{aligned} \tag{4.4}$$

are, respectively the creation and annihilation operators of the boson pairs. They are the invariant quantities of group O(N), i.e. they commutate with the generators of group O(N). Therefore, we see that $p_u^\dagger$ and $\tilde{p}_u$ commutate with the Casimir operator of O(N):

$$\begin{aligned} &[C_{2ON}, p_u^\dagger]=0 \\ &[C_{2ON}, \tilde{p}_u]=0. \end{aligned} \tag{4.5}$$

Consider the wavefunctions of v bosons $|vv\,\alpha\,Lf\rangle$ which satisfy the following equations:

$$\begin{bmatrix} \hat{n}_b \\ \hat{L}^2 \\ \hat{F}^2 \\ \tilde{p}_u \end{bmatrix} |vv\,\alpha\,Lf\rangle = \begin{bmatrix} v \\ L(L+1) \\ f(f+1) \\ 0 \end{bmatrix} \quad |vv\,\alpha\,Lf\rangle \tag{4.6}$$

From equation (4.6) we see that any two of these v bosons cannot construct a pair. Because $p_u^\dagger$ is the invariant quantity of group O(N), $|n[n_1n_2]\,(\sigma_1\sigma_2)\,\alpha\,LF\rangle$ can be obtained by means of $p_u^\dagger$ acting step by step on $|vv\,\alpha\,Lf\rangle$, i.e.

$$\begin{aligned} |n[n_1\,n_2]\,(\sigma_1\sigma_2)\,LF\rangle &= |n\,vf\,\alpha\,L\,F\rangle \\ &= C(n_1n_2\,\beta\,\sigma_1\sigma_2)S^{\dagger\beta}\{p_u f_u^{\dagger\rho}|\,vv\,\alpha\,Lf\rangle^F \end{aligned}$$

where

$$
\begin{aligned}
& v=\sigma_1+\sigma_2 \\
& f=\frac{\sigma_1-\sigma_2}{2} \\
& n=n_1+n_2=4\beta+2\rho+v \\
& F=\frac{n_1-n_2}{2}
\end{aligned}
\tag{4.7a}
$$

$$
\begin{aligned}
& S^\dagger=p^\dagger \cdot p^\dagger \\
& p_u^{\dagger\rho}=\{p^\dagger \times p^\dagger \times \ldots \times p^\dagger\}_u^\rho
\end{aligned}
\tag{4.7b}
$$

and are ρ factors of $p^\dagger$.

It is easy to show that $|n[n_1\, n_2]\beta(\sigma_1\sigma_2)\ \alpha\ L\ F\rangle$ satisfies equation (2.4). The wavefunctions (4.7) are the complete set of n boson wavefunctions, but they are not orthogonal to each other. The above discussion shows v is the seniority number, and f can be called the reduced F-spin, i.e. the F-spin of the v unpaired bosons. The normalization constant $C(n_1 n_2\ \beta\ \sigma_1\sigma_2)$ can be calculated for $\beta=1$ in the usual way, and the results are

$$
\begin{aligned}
& C(n_1\sigma_2 1\,\sigma_1\sigma_2)=\sqrt{\frac{\rho!\,(N+2\sigma_1+2\rho-2)!!}{(N+2\sigma_1-2)!!}} \\
& C(n_1\sigma_2+1\ 1\sigma_1\sigma_2)=\sqrt{\frac{\rho!\,(N+2\sigma_1+2\rho-4)!!\,(N+\sigma_1+\sigma_2-2)}{(N+2\sigma_1-2)!!}}.
\end{aligned}
\tag{4.8}
$$

The wavefunction $|\,v v L_{\max} f\rangle$ is the highest-weight state (HWS) of the IRRP $(\sigma_1\sigma_2)$ of group $O(N)$. It can be calculated as

$$
|(\sigma_1\sigma_2)\,\text{HWS}\rangle=|\,v v L_{\max} f\rangle=\frac{(\sigma_1-\sigma_2+1)!}{(\sigma_1+1)!\,\sigma_2!}B^{\dagger\sigma_2}b_{l\frac{1}{2}}^{\dagger(\sigma_1-\sigma_2)}|0\rangle \tag{4.9a}
$$

where

$$
B^\dagger=\{b^\dagger b^\dagger\}_{2l-1\,0}^{2l-1\,0}=b_{l\frac{1}{2}}^\dagger b_{l-1\,-\frac{1}{2}}^\dagger-b_{l-1\,\frac{1}{2}}^\dagger b_{l\,-\frac{1}{2}}^\dagger. \tag{4.9b}
$$

Using equation (4.9*a*) we can obtain the HWS† of the IRRP n of group $U(2N)$ and that of IRRP $[n_1, n_2]$ of group $U(N)$. They are

$$
|n\ \text{HWS}\rangle=\sqrt{\frac{1}{n!}}\,b_{l\,1/2}^{\dagger n}|0\rangle \tag{4.10}
$$

$$
|[n_1, n_2]\ \text{HWS}\rangle=|(n_1\ n_2)\,\text{HWS}\rangle=\frac{(n_1-n_2+1)!}{(n_1+1)!\,n_2!}B^{\dagger n_2}b_{l\,1/2}^{\dagger(n_1-n_2)}|0\rangle. \tag{4.11}
$$

† The HWS is a simple state, and we omit the additional quantum number α as usual.

5. The CFP

5.1. Factorization

The CFP for the bosons whose wavefunctions can be labelled by the group chain (2.1) can be expressed in the second quantization representation as

$$\langle nvf\alpha L\beta F\{|n-1\ v'f'\alpha'L'\beta'F'\rangle=\sqrt{\frac{1}{n}}\langle nvf\alpha L\beta F\|b^\dagger\|n-1\ v'f'\alpha'L'\beta'F'\rangle \tag{5.1}$$

where $\langle\ldots\|b^\dagger\|\ldots\rangle$ is the reduced matrix element† of the irreducible tensor operator $b^\dagger_{m\sigma}$ [28]. It is easy to show that $b^\dagger_{m\sigma}$ is the irreducible tensor with rank 1 under the group chain (2.1), i.e.

$$\begin{aligned} b^\dagger_{m\sigma}=b^\dagger(1\,[1]\,(1\ 0)\ l\ m\ \tfrac{1}{2}\sigma)& \\ \mathrm{U}(2N)\ \mathrm{U}(N)\ \mathrm{O}(N)\ \mathrm{O}(3)\ \mathrm{SU}(2).& \end{aligned} \tag{5.2}$$

Then, according to the generalized Wigner–Eckart theorem† the reduced matrix element of $b^\dagger_{m\sigma}$ can be factorized, with respect to the group chain (2.1), as the following:

$$\begin{aligned}\langle nvf\alpha L\beta F\|b^\dagger\|n-1\,v'f'\alpha'L'\beta'F'\rangle=\sqrt{n}&\left[\begin{array}{cc|c}1 & n-1 & n\\ {[1]} & [n_1'n_2'] & [n_1n_2]\end{array}\right]\\ &\left[\begin{array}{cc|c}{[1]} & [n_1'n_2'] & [n_1n_2]\\ (1) & \beta'(\sigma_1'\sigma_2') & \beta(\sigma_1\sigma_2)\end{array}\right]\left[\begin{array}{cc|c}(1) & (\sigma_1'\sigma_2') & (\sigma_1\sigma_2)\\ l & \alpha'L' & \alpha L\end{array}\right]\end{aligned} \tag{5.3}$$

where

$$\left[\begin{array}{cc|c}1 & n-1 & n\\ {[1]} & [n'n'] & [n\quad n]\end{array}\right]$$

is the isoscalar factor of $\mathrm{U}(2N)\supset\mathrm{U}(N)$,

$$\left[\begin{array}{cc|c}{[1]} & [n_1'n_2'] & [n_1n_2]\\ (1) & \beta'(\sigma_1'\sigma_2') & \beta(\sigma_1\sigma_2)\end{array}\right]$$

is the isoscalar factor of $\mathrm{U}(N)\supset\mathrm{O}(N)$, and

$$\left[\begin{array}{cc|c}(1) & (\sigma_1'\sigma_2') & (\sigma_1\sigma_2)\\ l & \alpha'L' & \alpha L\end{array}\right]$$

† Here we use the formula

$$\langle LF\|b^\dagger\|L'F'\rangle=\sum_{\substack{mM'\\ \sigma K'}}\langle LMFK|b^\dagger_{m\sigma}|L'M'F'K'\rangle\langle lmL'M'|LM\rangle\langle\tfrac{1}{2}\sigma F'K'|FK\rangle.$$

Table 3. The isoscalar factors of $U(2N) \supset U(N)$.

$$\left[\begin{array}{cc|c} 1 & n-1 & n \\ [1] & [n_1-1\; n_2] & [n_1 n_2] \end{array}\right] = \sqrt{\frac{(n_1+1)(n_1-n_2)}{(n_1+n_2)(n_1-n_2+1)}}$$

$$\left[\begin{array}{cc|c} 1 & n-1 & n \\ [1] & [n_1 n_2-1] & [n_1 n_2] \end{array}\right] = \sqrt{\frac{n_2(n_1-n_2+2)}{(n_1+n_2)(n_1-n_2+1)}}$$

Table 4. The isoscalar factors of $U(N) \supset O(N)$.

$$\left[\begin{array}{cc|c} [1] & [n-1] & [n] \\ (1) & (\sigma-1) & (\sigma) \end{array}\right] = \sqrt{\frac{\sigma(N+n+\sigma-2)}{n(N+2\sigma-2)}}$$

$$\left[\begin{array}{cc|c} [1] & [n-1] & [n] \\ (1) & (\sigma+1) & (\sigma) \end{array}\right] = \sqrt{\frac{(n-\sigma)(N+\sigma-2)}{n(N+2\sigma-2)}}$$

$$\left[\begin{array}{cc|c} [1] & [n-1] & [n-1\;1] \\ (1) & (\sigma-1) & (\sigma) \end{array}\right] = \sqrt{\frac{(n-\sigma)(N+\sigma-2)}{n(N+2\sigma-2)}}$$

$$\left[\begin{array}{cc|c} [1] & [n-1] & [n+1] \\ (1) & (\sigma+1) & (\sigma) \end{array}\right] = \sqrt{\frac{\sigma(N+n+\sigma-2)}{n(N+2\sigma-2)}}$$

$$\left[\begin{array}{cc|c} [1] & [n-1] & [n-1\;1] \\ (1) & (\sigma) & (\sigma 1) \end{array}\right] = 1$$

$$\left[\begin{array}{cc|c} [1] & [n-2\;1] & [n-1\;1] \\ (1) & (\sigma-1) & (\sigma) \end{array}\right] = \sqrt{\frac{(\sigma-1)(N+\sigma-2)(N+n+\sigma-4)}{(n-2)(N+\sigma-3)(N+2\sigma-2)}}$$

$$\left[\begin{array}{cc|c} [1] & [n-2\;1] & [n-1\;1] \\ (1) & (\sigma+1) & (\sigma) \end{array}\right] = \sqrt{\frac{\sigma(n-\sigma-2)(N+\sigma-1)}{(n-2)(\sigma+1)(N+2\sigma-2)}}$$

$$\left[\begin{array}{cc|c} [1] & [n-2\;1] & [n-1\;1] \\ (1) & (\sigma 1) & (\sigma) \end{array}\right] = \sqrt{\frac{(n-1)(N-2)}{(n-2)(\sigma+1)(N+\sigma-3)}}$$

$$\left[\begin{array}{cc|c} [1] & [N-2\;1] & [n-1\;1] \\ (1) & (\sigma-1\;1) & (\sigma 1) \end{array}\right] = \sqrt{\frac{(n-1)(\sigma+1)(\sigma-1)(N+n+\sigma-3)}{n(n-2)\sigma(N+2\sigma-2)}}$$

$$\left[\begin{array}{cc|c} [1] & [n-2\;1] & [n-1\;1] \\ (1) & (\sigma+1\;1) & (\sigma 1) \end{array}\right] = \sqrt{\frac{(n-1)(n-\sigma-1)(N+\sigma-1)(N+\sigma-3)}{n(n-2)(N+\sigma-2)(N+2\sigma-2)}}$$

$$\left[\begin{array}{cc|c} [1] & [n-2\;1] & [n-1\;1] \\ (1) & (\sigma) & (\sigma 1) \end{array}\right] = \sqrt{\frac{(n-\sigma-1)(N+n+\sigma-3)}{n(n-2)\sigma(N+\sigma-2)}}$$

is the isoscalar factor of $O(N) \supset O(3)$. The isoscalar factors of $U(2N) \supset U(N)$ and $U(N) \supset O(N)$ can be obtained by using the HWS of the IRRP $[n_1 n_2]$ of group $U(N)$ and the IRRP $(\sigma_1 \sigma_2)$ of $O(N)$. The results are shown in tables 3 and 4.

Table 5. Coefficients $\langle n v f B F\{| n-1\, v' f' \beta' F'\rangle$.

v'	f'	F'	$(v'fF)$ $v\frac{v}{2}\frac{n}{2}$	(vfF) $v\frac{v}{2}\frac{n-2}{2}$	(vfF) $v\frac{v-2}{2}\frac{n-2}{2}$
$v-1$	$f-\frac{1}{2}$	$F-\frac{1}{2}$	$\sqrt{\frac{v(N+n+v-2)}{n(N+2v-2)}}$	$\sqrt{\frac{(v-1)(N+v-2)(N+n+v-4)}{(n-1)(N+v-3)(N+2v-2)}}$	$\sqrt{\frac{v(v-2)(N+n+v-3)}{n(v-1)(N+2v-4)}}$
$v-1$	$f+\frac{1}{2}$	$F-\frac{1}{2}$			$\sqrt{\frac{(n-v)(N+n+v-4)}{n(n-1)(v-1)(N+v-3)}}$
$v+1$	$f-\frac{1}{2}$	$F-\frac{1}{2}$		$\sqrt{\frac{(N-2)}{(v+1)(N+v-3)}}$	
$v+1$	$f+\frac{1}{2}$	$F-\frac{1}{2}$	$\sqrt{\frac{(n-v)(N+v-2)}{n(n+2v-2)}}$	$\sqrt{\frac{v(n-v-2)(N+v-1)}{(n-1)(\sigma+1)(N+2\sigma-1)}}$	$\sqrt{\frac{(n-v)(N+v-2)(N+v-4)}{n(n+v-3)(N+2v-4)}}$
$v-1$	$f-\frac{1}{2}$	$F+\frac{1}{2}$		$\sqrt{-\frac{(n-v)(n+v-2)}{(n-1)(n+2v-2)}}$	
$v-1$	$f+\frac{1}{2}$	$F+\frac{1}{2}$			$\sqrt{\frac{1}{n-1}}$
$v+1$	$f-\frac{1}{2}$	$F+\frac{1}{2}$			
$v+1$	$f+\frac{1}{2}$	$F+\frac{1}{2}$		$\sqrt{\frac{v(N+n+v-2)}{n(n-1)(n+2v-2)}}$	

The CFP shown in equation (5.1) can then be rewritten as

$$\langle n v f \alpha L \beta F\{| n-1\ v' f' \alpha' L' \beta' F'\rangle = \langle n v f \beta F\{| n-1\ v' f' \beta' f'\rangle \langle v \alpha L f\{| v' \alpha' L' f'\rangle \tag{5.4}$$

where

$$\langle n v f \beta F\{| n-1\, v' f' \beta' F'\rangle = \left[\begin{array}{cc|c} 1 & n-1 & n \\ [1] & [n_1 n_2] & [n_1 n_2] \end{array}\right] \left[\begin{array}{cc|c} [1] & [n_1' n_2'] & [n_1 n_2] \\ (1) & \beta'(\sigma_1 \sigma_2) & \beta(\sigma_1 \sigma_2) \end{array}\right] \tag{5.5}$$

can be designated as the F-spin part of the CFP, and

$$\langle v \alpha L f\{| v' \alpha' L' f'\rangle = \left[\begin{array}{cc|c} (1) & (\sigma_1' \sigma_2') & (\sigma_1 \sigma_2) \\ l & \alpha' L' & \alpha L \end{array}\right] \tag{5.6}$$

can be referred to as the angular momentum part of the CFP. The values of $\langle n v f \beta F\{| n-1\ v' f' \beta' F'\rangle$ with $F=n/2$ and $(n-2)/2$ are given in table 5.

5.2. *Reciprocal relation*

It is easy to show that the boson operators $b_{m\sigma}^{\dagger}$ and $\tilde{b}_{m\sigma}$ are the $(1)\times 1/2$ rank tensor operators of group chain $(\mathrm{O}(N)\supset \mathrm{O}(3))\times \mathrm{SU}(2)$, i.e.

$$b_{m\sigma}^{\dagger} = b^{\dagger}\ \begin{array}{ccc}((1) & lm & \frac{1}{2}\sigma)) \\ \mathrm{O}(N) & \mathrm{O}(3) & \mathrm{SU}(2) \end{array} \tag{5.7}$$

$$\tilde{b}_{m\sigma} = \tilde{b}\ \begin{array}{ccc}((1) & lm & \frac{1}{2}\sigma)) \\ \mathrm{O}(N) & \mathrm{O}(3) & \mathrm{SU}(2). \end{array} \tag{5.8}$$

According to the generalized Wigner–Eckart theorem, we have

$$\langle(\sigma_1\sigma_2)\alpha Lf\|b^\dagger\|(\sigma_1\sigma_2)\alpha' L'f'\rangle=\langle(\sigma_1\sigma_2)f\|b^\dagger\|(\sigma'_1\sigma'_2)f'\rangle\begin{bmatrix}(1) & (\sigma_1\sigma_2) & \big| & (\sigma'_1\sigma'_2)\\ l & \alpha' L' & \big| & \alpha L\end{bmatrix}$$

and

$$\langle(\sigma'_1\sigma'_2)\alpha' L'f''\|\tilde{b}\|(\sigma_1\sigma_2)\alpha Lf\rangle$$
$$=\langle(\sigma'_1\sigma'_2)f'\|\tilde{b}\|(\sigma_1\sigma_2)f\rangle\begin{bmatrix}(1) & (\sigma_1\sigma_2) & \big| & (\sigma'_1\sigma'_2)\\ l & \alpha L & \big| & \alpha' L'\end{bmatrix} \tag{5.10a}$$

where

$$(\sigma'_1\sigma'_2)=(\sigma_1-1\ \sigma_2)\ \text{or}\ (\sigma_1\sigma_2-1) \tag{5.10b}$$

and $\langle(\sigma_1\sigma_2)f\|b^\dagger\|(\sigma'_1\sigma'_2)f'\rangle$, $\langle(\sigma'_1\sigma'_2)f'\|\tilde{b}\|(\sigma'_1\sigma'_2)f\rangle$ are the reduced matrix elements of group $O(N)\times SU(2)$. Using the relation

$$\sqrt{(2L+1)(2f+1)}\,\langle(\sigma_1\sigma_2)\alpha Lf\|b^\dagger\|(\sigma'_1\sigma'_2)\alpha' L'f'\rangle$$
$$=(-)^{L+l-L'}(-)^{f+1/2-f'}\sqrt{(2L+1)(2f'+1)}$$
$$\times\langle(\sigma'_1\sigma'_2)\alpha' L'f'\|\tilde{b}\|(\sigma_1\sigma_2)\alpha Lf\rangle \tag{5.11}$$

and the normalization of the isoscalar factors we obtain

$$\langle(\sigma'_1\sigma'_2)f'\|\tilde{b}\|(\sigma_1\sigma_2)f\rangle$$
$$=(-)^{f+1/2-f'}\sqrt{\frac{d(\sigma_1\sigma_2)\,(2f+1)}{d(\sigma'_1\sigma'_2)\,(2f'+1)}}\langle(\sigma_1\sigma_2)f'\|b^\dagger\|(\sigma'_1\sigma'_2)f'\rangle \tag{5.12}$$

$$\begin{bmatrix}(1) & (\sigma_1\sigma_2) & \big| & (\sigma'_1\sigma'_2)\\ l & \alpha L & \big| & \alpha' L'\end{bmatrix}$$
$$=(-)^{l+L-L'}\sqrt{\frac{d(\sigma'_1\sigma'_2)\,(2L+1)}{d(\sigma_1\sigma_2)\,(2L'+1)}}\begin{bmatrix}(1) & (\sigma'_1(\sigma'_2) & \big| & (\sigma_1\sigma_2)\\ l & \alpha' L' & \big| & \alpha L\end{bmatrix}. \tag{5.13}$$

Equation (5.13) is called the reciprocal rule of the isoscalar factor of $O(N)\supset O(3)$.

5.3. The recursion relation

From the above discussion we see that the isoscalar factors $O(N)\supset O(3)$ satisfy the following equation:

$$\langle(\sigma_1\sigma_2)f\|b^\dagger\|(\sigma'_1\sigma'_2)f'\rangle\begin{bmatrix}(1) & (\sigma_1\sigma_2) & \big| & (\sigma_1\sigma_2)\\ l & \alpha' L' & \big| & \alpha L\end{bmatrix}=\langle v\alpha Lf\|b^\dagger\|v-1\alpha' L'f'\rangle. \tag{5.14}$$

Suppose the wavefunctions $|v-1\alpha' L'f'\rangle, |v-2\alpha'' L''f''\rangle,\ldots$ are known. We then define the states

$$|\Psi(v[\alpha'_1L'_1f'_1]Lf\rangle=\{b^\dagger|v-1\ \alpha'_1L'_1f'_1\rangle\}^{Lf}. \tag{5.15}$$

The states $|\Psi(v[\alpha_1' L_1' f_1']Lf)\rangle$ have boson number $n=v$ and seniority equal to either v or $v-2$. Therefore, they can be expanded as

$$|\Psi(v[\alpha_1' L_1' f_1']Lf\rangle = A(v[\alpha_1' L_1' f_1']Lf)|v[\alpha_1' L_1' f_1']Lf\rangle + \sum_{\alpha'' f''} B(v[\alpha_1' L_1' f_1']Lf\alpha'' L'')|v\ v-2\ \alpha'' f'' Lf\rangle \tag{5.16}$$

where $|v[\alpha_1' L_1' f_1'\rangle$ means a state with definite seniority v and reduced F-spin f. The additional quantum number $[\alpha_1' L_1' f_1']$, means that the states are obtained from the parent states $|\Psi(v[\alpha_1' L_1' f_1']Lf)\rangle$:

$$|v[\alpha_1' L_1' f_1']Lf\rangle = \frac{1}{A(v[\alpha_1' L_1' f_1']Lf)}\{|\Psi(v[\alpha_1' L_1' f_1']Lf)\rangle - \sum_{\alpha'' f''} B(v[\alpha_1' L_1' f_1']Lf\alpha'' f'')|v\ v-2\ \alpha'' f'' Lf\rangle\}. \tag{5.17}$$

From the orthogonality of the wavefunctions with different seniority or reduced F-spin and by using the commutator of the pair annihilation operators p_u and pair creation operators $p_u^\dagger$, we get

$$\begin{aligned} &B(v[\alpha' L' f']Lf\alpha'' f'') \\ &= \langle v\ v-2\ \alpha'' f''\ Lf\|b^\dagger\|v-1\ v-1\ \alpha_1'\ L_1'\ F_1'\rangle \\ &= (-)^{l+L+L'+f+f''+1}\sqrt{\frac{6(2L_1'+1)(2f_1'+1)}{(2L+1)(N+v-4+f(f+1)-f''(f''+1)}} \\ &\times \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ f'' & f & f_1' \end{Bmatrix} \langle v-1\ v-1\ \alpha_1' L_1' f_1'\|b^+\|v-2\ v-2\ \alpha'' L'' f''\rangle. \end{aligned} \tag{5.18}$$

Let

$$\begin{aligned} &R(v[\alpha_1' L_1' f_1']\alpha' L' f' Lf) \\ &= \delta(\alpha_1'\alpha')\delta(L_1' L')\delta(f_1' f) - (-)^{L_1'+L'+f_1'+f'} \\ &\times \sqrt{(2L_1'+1)(2L'+1)(2f_1'+1)(2f'+1)}. \\ &\times \sum_{\alpha'' L'' f''}\left[\begin{Bmatrix} l & L'' & L_1' \\ l & L & L' \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & f'' & f_1' \\ \frac{1}{2} & f & f' \end{Bmatrix}\right. \\ &+ \frac{(-)^{f'+f_1'}6\delta(LL'')}{(2L+1)[N+v-4+f(f+1)-f''(f''+1)]} \\ &\times \left.\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ f'' & f & f' \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ f'' & f & f' \end{Bmatrix}\right]\langle v-1\ v-1\ \alpha_1' f_1' L_1'\|b^\dagger\|v-2\ v-2\ \alpha'' L'' f''\rangle \\ &\times \langle v-1\ v-1\ \alpha' L' f'\|b^\dagger\|v-2\ v-2\ \alpha'' L'' f''\rangle. \end{aligned} \tag{5.19}$$

From equations (5.17), (5.18) and (5.19) we get

$$A(v[\alpha' L' f']Lf) = \sqrt{R(v[\alpha_1' L_1' f_1']\alpha_1' L_1' f_1'\ Lf)}. \tag{5.20}$$

The recurrent formula for the reduced matrix element is

$$\langle vv[\alpha_1' L_1' f_1']Lf \| b^{\dagger} \| v-1\ v-1\ \alpha_1' L_1' f_1' \rangle$$
$$= R(v[\alpha_1' L_1' f_1']\alpha' L' f' Lf)/\sqrt{R(v[\alpha_1' L_1' f_1']\alpha_1' L_1' f_1' Lf)}.$$

From equations (5.14) and (5.21) we can obtain the O(N) ⊃ O(3) isoscalar factors

$$\left[\begin{matrix} (1) & (\sigma_1\sigma_2) \\ l & \alpha' L' \end{matrix} \middle| \begin{matrix} (\sigma_1\sigma_2) \\ [\alpha' L' f']L \end{matrix}\right].$$

Of course, the states

$$\left| \begin{matrix} (\alpha_1\sigma_2) \\ [\alpha' L' f']L \end{matrix} \right\rangle$$

are overcomplete and non-orthogonal. We can use an orthogonalization method, such as the Schmidt method, to obtain the orthonormal complete states

$$\left| \begin{matrix} (\sigma_1\sigma_2) \\ \alpha L \end{matrix} \right\rangle$$

and the corresponding isoscalar factors

$$\left[\begin{matrix} (1) & (\sigma_1'\sigma_1') \\ l & \alpha' L' \end{matrix} \middle| \begin{matrix} (\sigma_1\sigma_2) \\ \alpha L \end{matrix}\right].$$

In order to make the orthogonal process more efficient, we use the multiplicity to control it [29]. We begin the recurrence process with two bosons with a state of seniority of two. The initial result of O(N) ⊃ O(3) isoscalar factors can be obtained by straight forward calculations:

$$\left[\begin{matrix} (1) & (1) \\ l & l \end{matrix} \middle| \begin{matrix} (20) \\ L \end{matrix}\right] = 1 \qquad L = 0, 2, 4, \ldots, 2l$$
$$\left[\begin{matrix} (1) & (1) \\ l & l \end{matrix} \middle| \begin{matrix} (11) \\ L \end{matrix}\right] = 1 \qquad L = 1, 3, 5, \ldots, 2l-1. \tag{5.21}$$

From equations (5.21), using recurrence and orthogonalization repeatedly, we can obtain any isoscalar factors of O(N) ⊃ O(3) numerically.

6. Conclusion

In summary, the classification of the wavefunctions of the boson system with single angular momentum l and F-spin $\frac{1}{2}$ and the branching rules of IRRP reductions in the group chain U(2N) ⊃ (U(N) ⊃ O(N) ⊃ O(3)) × SU(2) are discussed. The CFP for the boson system is factorized into the F-spin part $\langle nvf\beta F\{|n-1\ v'f'v'F'\rangle$ and the angular momentum part $\langle sf\alpha L\{|s'f'\alpha' L'\rangle$. An approach to evaluate these factors is provided. A computer code based on the formulae has also been set up to determine the CFP of IBM2 [29]. We hope that this discussion will facilitate the IBM calculations and promote further discussions on superdeformed nuclei and the chaotic behaviour of many-boson systems.

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